Kinematic Model for Fixed-Wing Aircraft with Constrained Roll-Rate

Mirco Theile and Simon Yu

September 15, 2018

1 Introduction

The magnitude of the roll rate for a fixed-wing aircraft is limited. For some systems such as the aerobatic aircraft, the maximum roll rate can be very high, allowing them to perform a change of roll angle that can be considered almost instantaneous. The assumption, on the other hand, cannot be applied to some other systems, like non-acrobatic airplanes and aircraft with passengers or sensitive equipment on board. Therefore, a kinematic model for an aircraft should not only take the maximum roll angle into account but also the maximum roll rate.

If the aircraft's maximum roll rate is limited, it is conceivable that a constant roll rate that is close to the limit would apply to its rolling maneuvers. As a result, based on the constant roll rate assumption, this technical report derives a kinematic model that can be applied to the trajectory planning of a fixed-wing aircraft. The model only considers constant altitude maneuvers, meaning that the lift of the aircraft balances the roll related rotation of the weight, leading to a predictable yaw rate depending on the momentary roll angle. Furthermore, the velocity of the aircraft is assumed to be constant, and the effects of wind are neglected. In summary, the assumptions are the following:

1. Constant roll rate.
2. Constant velocity.
3. Constant altitude.
4. Wind effects neglected.

The above assumptions will be addressed in future work, as the current goal only focuses on showing the derivation and the applicability of the kinematic model despite those limitations.

2 Derivation of the Kinematic Model

The following section contains the derivation of the kinematic model that estimates a constant roll rate trajectory. For the derivation, the altitude of the aircraft is ignored. As a result, the position of the aircraft in x and y can be described in the complex plane as $c(t) = x(t) + \mathrm{i}y(t)$. Correspondingly, the velocity of the aircraft can also be expressed as $v(t) = v_x(t) + \mathrm{i}v_y(t)$. The main equations from which the model is derived are the following:

$$\phi(t) = rt + \phi_0 \tag{2.1}$$

with t being time, r being the constant roll rate, and ϕ_0 being the initial roll angle. From the roll angle, the curvature $\dot{\psi}$, namely the yaw rate, can be expressed as follows:

$$\dot{\psi}(t) = -\frac{g}{V} \tan \phi(t) \tag{2.2}$$

with g being the gravitational constant, and V being the constant total velocity. The above relation is based on the lift compensation for the rotated weight vector due to a rolling maneuver. Consequently, the yaw angle of the aircraft is defined as

$$\psi(t) = \int_0^t \dot{\psi}(u) du + \psi_0 \tag{2.3}$$

with ψ_0 being the initial yaw angle of the aircraft. Using the yaw angle, the velocity can be described as

$$v(t) = V (\cos \psi(t) + \mathrm{i} \sin \psi(t)) \tag{2.4}$$

which is defined using Euler's Equation as

$$v(t) = V \mathrm{e}^{\mathrm{i}\psi(t)} \tag{2.5}$$

From the above equation, the position can be expressed as

$$c(t) = \int_0^t v(u) du + c_0 \tag{2.6}$$

with the initial position being c_0.

From this point onward, the derivation of the model is split into two cases, a simplified case and the general case. For the simplified case, we define the initial roll angle of the aircraft as zero degrees which will be later generalized for all initial roll angles in the general case.

2.1 Simplified Case

The derivation of the kinematic model starts with a simplified case where a few constraints are applied to the calculation. We first assume that the current roll angle of the aircraft is zero, i.e., the plane is flying with perfectly leveled wings. In addition, we assume that the current yaw angle, as well as the current position of the aircraft are also zero.

The constraints can be defined as the following, where ϕ_0 is the current roll angle, ψ_0 is the current yaw angle, and c_0 is the current position of the aircraft:

$$\phi_0 = 0, \tag{2.7}$$
$$\psi_0 = 0, \tag{2.8}$$
$$c_0 = 0 + \mathrm{i}0 \tag{2.9}$$

We define the roll angle of the aircraft ϕ at any point of time t given a constant roll rate r to be a linear function as the following:

$$\phi(t) = rt \tag{2.10}$$

Using the roll angle ϕ, we can then express the curvature of the trajectory of the aircraft at each time point to be the following function:

$$\dot{\psi}(t) = -\frac{g}{V}\tan\phi(t) \tag{2.11}$$

By integrating the curvature function, we can obtain the yaw angle of the aircraft with respect to time:

$$\psi(t) = -\frac{g}{V}\int_0^t \tan\phi(u)du$$

from which we can write

$$\psi(t) = \frac{g}{rV}\ln|\cos\phi(t)| \tag{2.12}$$

Since $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the cosine is always positive, which simplifies the above equation to

$$\psi(t) = \frac{g}{rV}\ln\cos\phi(t) \tag{2.13}$$

With the yaw angle function, we can write the ground velocity of the aircraft as a function of time in the following complex form:

$$v(t) = V\mathrm{e}^{\mathrm{i}\psi(t)} = V\mathrm{e}^{\mathrm{i}\frac{g}{rV}\ln\cos\phi(t)} \tag{2.14}$$

By integrating the velocity function, we obtain the following function that represents the position of the aircraft as the function of time which, when solved, represents the kinematic model of the aircraft:

$$c(t) = V\int_0^t \mathrm{e}^{\mathrm{i}\frac{g}{rV}\ln\cos\phi(u)}du$$

Before solving the integral, we substitute time for roll angle such that the model is a function of roll angle:

$$\begin{aligned} \gamma &= \phi(u) \\ d\gamma &= d\phi(u)du = rdu \\ du &= \frac{1}{r}d\gamma \end{aligned}$$

from the above U-Substitution, we have

$$c(\phi) = \frac{V}{r}\int_0^\phi \mathrm{e}^{\mathrm{i}\frac{g}{rV}\ln\cos(\gamma)}d\gamma \tag{2.15}$$

Apply U-Substitution on $\ln\cos\gamma$:

$$\begin{aligned} w &= \ln\cos\gamma \\ dw &= -\tan\gamma d\gamma \\ \gamma &= \pm\cos^{-1}e^w \end{aligned}$$

In the above integral, $\gamma \in [0, \phi]$. Therefore, γ has the same sign as that of ϕ. Hence, we can write [1]

$$\gamma = \operatorname{sign}(\phi) \cos^{-1} e^w$$
$$d\gamma = -\frac{1}{\tan\gamma} dw = -\operatorname{sign}(\phi) \frac{1}{\tan\cos^{-1} e^w} dw = -\operatorname{sign}(\phi) \frac{e^w}{\sqrt{1 - \mathrm{e}^{2w}}} dw$$

Applying the above substitution yields the following:

$$c(\phi) = -\operatorname{sign}(\phi) \frac{V}{r} \int_0^{\ln\cos\phi} \mathrm{e}^{(1+\mathrm{i}\frac{g}{Vr})w} \left(1 - \mathrm{e}^{2w}\right)^{-\frac{1}{2}} dw$$

The form of the equation closely resembles the Incomplete Beta Function which is defined as the following:

$$B(x; a, b) = \int_0^x y^{a-1}(1-y)^{b-1} dy \tag{2.16}$$

As a result, we can perform another U-Substitution on e^{2w} such that the equation satisfies the form of the Incomplete Beta Function:

$$y = \mathrm{e}^{2w}$$
$$dy = 2y dw$$

Applying the above substitution yields the following:

$$c(\phi) = -\operatorname{sign}(\phi) \frac{V}{2r} \int_1^{\cos^2\phi} y^{(-\frac{1}{2}+\mathrm{i}\frac{g}{2Vr})} (1-y)^{-\frac{1}{2}} dy = \operatorname{sign}(\phi) \frac{V}{2r} \int_{\cos^2\phi}^1 y^{a-1}(1-y)^{b-1} dy$$

According to the definition of the Incomplete Beta function and the form of the above equation, the parameters a, b for the Incomplete Beta Function can be defined as the following:

$$a = \frac{1}{2} + \mathrm{i}\frac{g}{2Vr}, \quad b = \frac{1}{2} \tag{2.17}$$

And in order to satisfy the integral bound of the Incomplete Beta Function, we can decompose the integral into the following form:

$$\begin{aligned} c(\phi) &= \operatorname{sign}(\phi) \frac{V}{2r} \int_{\cos^2\phi}^1 y^{a-1}(1-y)^{b-1} dy \\ &= \operatorname{sign}(\phi) \frac{V}{2r} \left[\int_0^1 y^{a-1}(1-y)^{b-1} dy - \int_0^{\cos^2\phi} y^{a-1}(1-y)^{b-1} dy \right] \end{aligned}$$

from which we can conclude:

$$c(\phi) = \operatorname{sign}(\phi) \frac{V}{2r} \left[B(1; a, b) - B(\cos^2\phi; a, b) \right] \tag{2.18}$$

[1] $\tan\cos^{-1} x = \frac{\sqrt{1-x^2}}{x}$

2.2 General Case

The simplified case is based on the current roll angle of the aircraft being zero. In general, however, the aircraft's roll angle ϕ_0 during real flights can have any value in $[-\phi_{max}, \phi_{max}]$, where ϕ_{max} is the maximum possible roll angle. To derive the general case, we can utilize the freedom of selecting the initial yaw angle ψ_0 and the initial position c_0 which can be considered as a change of reference frame.

We start the derivation for the general case by redefining the roll angle function which takes the initial roll angle ϕ_0 of the aircraft into consideration:

$$\phi(t) = rt + \phi_0 \tag{2.19}$$

Similar to the simplified case, we have the curvature function defined as the following:

$$\dot{\psi}(t) = -\frac{g}{V} \tan \phi(t) \tag{2.20}$$

integrating it yields the yaw angle function:

$$\psi(t) = -\frac{g}{V} \int_0^t \tan \phi(u) du + \psi_0$$

In contrast to the simplified case, the yaw angle in the general case evaluates to:

$$\psi(t) = \frac{g}{rV} (\ln \cos \phi(t) - \ln \cos \phi_0) + \psi_0$$

When solving the above integral, the goal here is to change the frame of reference such that $\psi(\phi = 0) = 0$. For $\phi = 0$, we have $\phi_0 + rt = 0$ which means $t = -\frac{\phi_0}{r}$. A negative time value represents a projection of the trajectory into the past that determines the center of our frame of reference where the roll angle and the yaw angle of the aircraft is zero. By substituting the time value into the yaw angle function

$$\psi(t = -\frac{\phi_0}{r}) = \frac{g}{rV} \left[\ln \cos \phi(-\frac{\phi_0}{r}) - \ln \cos \phi_0 \right] + \psi_0 = 0$$

solving for ψ_0 yields:

$$\psi_0 = \frac{g}{rV} \ln \cos \phi_0 \tag{2.21}$$

By transforming the frame of reference such that the aircraft's yaw angle at the time $t = 0$ is ψ_0, we can write

$$\psi(t) = \frac{g}{rV} \ln \cos \phi(t) \tag{2.22}$$

The similar change-of-frame strategy can later be applied for the position function, which is identical to the function in the simplified case except for the initial position c_0:

$$c(t) = V \int_0^t \mathrm{e}^{\mathrm{i}\psi(u)} du + c_0 = V \int_0^t \mathrm{e}^{\mathrm{i} \frac{g}{rV} \ln \cos \phi(u)} du + c_0$$

As in the simplified case, we substitute time for roll angle such that the trajectory model is a function of roll angle:

$$c(\phi) = \frac{V}{r}\int_{\phi_0}^{\phi} \mathrm{e}^{\mathrm{i}\frac{g}{rV}\ln\cos\gamma} d\gamma + c_0 \tag{2.23}$$

This equation for the position is now very similar to the simplified case in (2.15), which, combined with the end result of the simple case in (2.18), yields

$$\frac{V}{r}\int_{0}^{\phi} \mathrm{e}^{\mathrm{i}\frac{g}{rV}\ln\cos(\gamma)} d\gamma = \mathrm{sign}(\phi)\frac{V}{2r}\left[B(1;a,b) - B(\cos^2\phi;a,b)\right]$$

To use the above relation for (2.23), the integral is split at 0:

$$\begin{aligned}
c(\phi) =& \frac{V}{r}\left(\int_{\phi_0}^{0} \mathrm{e}^{\mathrm{i}\frac{g}{rV}\ln\cos\gamma} d\gamma + \int_{0}^{\phi} \mathrm{e}^{\mathrm{i}\frac{g}{rV}\ln\cos\gamma} d\gamma\right) + c_0 \\
=& \frac{V}{r}\left(\int_{0}^{\phi} \mathrm{e}^{\mathrm{i}\frac{g}{rV}\ln\cos\gamma} d\gamma - \int_{0}^{\phi_0} \mathrm{e}^{\mathrm{i}\frac{g}{rV}\ln\cos\gamma} d\gamma\right) + c_0 \\
=& \frac{V}{2r}\left(\mathrm{sign}(\phi)\left[B(1;a,b) - B(\cos^2\phi;a,b)\right] - \mathrm{sign}(\phi_0)\left[B(1;a,b) - B(\cos^2\phi_0;a,b)\right]\right) + c_0
\end{aligned}$$

Similar to the yaw angle, we change the frame of reference such that $c(\phi = 0) = 0 + \mathrm{i}0$. For $\phi = 0$, we again have $\phi_0 + rt = 0$ and then $t = -\frac{\phi_0}{r}$. By substituting the negative time value into the position function, we can solve for c_0 as follows:

$$c_0 = \mathrm{sign}(\phi_0)\frac{V}{2r}\left[B(1;a,b) - B(\cos^2\phi_0;a,b)\right] \tag{2.24}$$

from which we conclude

$$c(\phi) = \mathrm{sign}(\phi)\frac{V}{2r}\left[B(1;a,b) - B(\cos^2\phi;a,b)\right] \tag{2.25}$$

3 Summary

The kinematic model describing the trajectory of an aircraft as a function of its roll angle ϕ with constant roll rate r, constant altitude, and constant velocity is concluded by the following equations:

The position of the aircraft described in $\mathbb{C}$ is described as

$$c(\phi) = \mathrm{sign}(\phi)\frac{V}{2r}\left[B(1;a,b) - B(\cos^2\phi;a,b)\right] := \beta(\phi) \tag{3.1}$$

where the initial position is

$$c_0 = \mathrm{sign}(\phi_0)\frac{V}{2r}\left[B(1;a,b) - B(\cos^2\phi_0;a,b)\right] := \beta(\phi_0) \tag{3.2}$$

The yaw angle of the aircraft is

$$\psi(t) = \frac{g}{rV}\ln\cos\phi(t) \tag{3.3}$$

and its initial yaw angle is

$$\psi_0 = \frac{g}{rV}\ln\cos\phi_0. \tag{3.4}$$